



# Does the crossover from perturbative to nonperturbative physics in QCD become a phase transition at infinite $N$ ?

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## Abstract

We present numerical evidence that, in the planar limit, four-dimensional Euclidean Yang–Mills theory undergoes a phase transition on a finite symmetrical four-torus when the length of the sides  $l$  decreases to a critical value  $l_c$ . For  $l > l_c$  continuum reduction holds so that at leading order in  $N$ , there are no finite size effects in Wilson and Polyakov loops. This produces the exciting possibility of solving numerically for the meson sector of planar QCD at a cost substantially smaller than that of quenched  $SU(3)$ .

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## 1. Introduction

Non-Abelian gauge theories in four dimensions interact strongly at large distances and weakly at short distances. The major conceptual achievement of lattice gauge theory has been to show that the continuum limit contains both regimes and interpolates between them in a smooth manner. At present, only numerical simulations can bridge these two regimes.

It has been a long held hope [1] that the task would simplify at infinite number of colors  $N$ . In its most optimistic version, the hope is that somebody will come up with an analytic solution to the  $N = \infty$ , “planar”, limit at all scales, and that it also will be feasible to

compute  $\frac{1}{N}$  corrections. We are far from attaining this goal, but there has been recent progress on similar, albeit vastly more constrained theories [2], the most notable example being  $\mathcal{N} = 4$  SUSY YM. The exact explicit solution of the planar limit in this case is limited to extreme cases, the simplest among them being when the 't Hooft coupling ( $\lambda = g_{YM}^2 N$ ) is taken to infinity. The general case has been mapped into a two-dimensional field theoretic problem which is not yet fully understood. The  $\frac{1}{N}$  corrections will come from the interactions in IIB string theory expanded around an  $AdS_5 \times S_5$  background stabilized by a nontrivial RR flux. At large  $\lambda$  many answers can be obtained by relatively simple calculations of small perturbations around this supergravity background.

In pure YM, we do not have a free coupling constant: instead there is nontrivial scale dependence reflecting the breaking of conformal invariance. A sim-

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ple way to visualize the situation is to consider Bjorken's "femto-universe" [3], where one studies the Hamiltonian of QCD restricted to a small three-dimensional box of side  $l$ . Equivalently, one can not only shrink the system (which takes care of linear momentum) but also raise its temperature (which deals with frequencies). The latter situation is best described by Euclidean field theory on a four torus. A remnant of Lorentz invariance is preserved when the temperature and linear size of the torus are related so that in Euclidean space the four torus has sides of equal length. As  $l$  is varied, the bulk physics of the system is perturbative for  $l\Lambda_{\text{QCD}} \ll 1$  and nonperturbative for  $l\Lambda_{\text{QCD}} \gg 1$ . Thus, the role of the coupling is taken up by  $l$ .

In this Letter, we consider  $SU(N)$  pure gauge theory in the planar limit on a four-torus of side  $l$ . The most basic question is whether, as a function of  $l$ , this system undergoes any phase transitions at  $N = \infty$ . There can be such transitions because  $N$  equals infinity: for finite  $N$  all these transitions will be smoothed out into crossovers. It is possible that when the torus is taken to infinite four-volume in a specific way, the crossovers become transitions even at finite  $N$ . How the various transitions merge into a coherent picture is a question that we have only begun to explore.

Recent work in three dimensions [4] has shown that when  $l$  is decreased from infinity, there will be a transition at  $l = l_c$ . For  $l > l_c$ , the system realizes a continuum version of Eguchi–Kawai reduction [5], whose salient feature is that expectation values of arbitrary Wilson loops are exactly  $l$ -independent once  $N = \infty$ . This continuum Eguchi–Kawai reduction is a property that is natural for a system of free strings but more difficult to understand in field theoretical terms where, in order to erase the perturbative  $l$ -dependence, one is required to enlist an averaging over a moduli space of minima of the classical action. In this Letter, we show that essentially the same effects work in four dimensions as well.

## 2. General phase structure

We used a single plaquette Wilson action as the cost of simulation increases with the number of colors  $N$  as  $N^3$ , making the simplicity of the action a relevant resource consideration. We worked on tori whose sides

consist of  $L$  lattice sites. The physical side length is  $l = aL$ , where  $a$  is the lattice spacing.  $L$  was varied from 1 to 10. The parameter  $N$  was chosen to be a prime number, taking the values  $N = 23, 31, 37$ . The preference for prime  $N$  is because the transition we found has to do with  $Z(N)$  groups, and for prime  $N$ ,  $Z(N)$  does not have subgroups that could confuse the picture. In one case, we did a simulation at  $N = 27$  and found nothing unusual, so it is possible that the choice of prime  $N$  values was unnecessary:

$$S = \frac{\beta}{4N} \sum_{x, \mu \neq \nu} \text{Tr}[U_{\mu, \nu}(x) + U_{\mu, \nu}^\dagger(x)], \quad (1)$$

$$U_{\mu, \nu}(x) = U_\mu(x)U_\nu(x + \mu)U_\mu^\dagger(x + \nu)U_\nu^\dagger(x). \quad (2)$$

We define  $b = \frac{\beta}{N^2} = \frac{1}{\lambda}$  and take the large  $N$  limit with  $b$  held fixed. As usual,  $b$  determines the lattice spacing  $a$ . The lattice is a symmetric torus of side  $L$ . The gauge fields are periodic.  $x$  is a four component integer vector labeling the site, and  $\mu$  either labels a direction or denotes a unit vector in the  $\mu$  direction. The link matrices  $U_\mu(x)$  are in  $SU(N)$ .

There is a  $Z^4(N)$  symmetry under which

$$U_\mu(x) \rightarrow e^{2\pi i k_\mu / N} U_\mu(x) \quad (3)$$

for all  $x$  with  $x_\mu = c_\mu$ . The integers  $c_\mu$  are fixed, and the integers  $k_\mu$  label the elements of the group;  $c_\mu, k_\mu = 0, 1, \dots, L - 1$ . Changing the  $c_\mu$ 's amounts to a local gauge transformation.

Polyakov loops are denoted by  $P_\mu(x)$  and defined by

$$P_\mu(x) = U_\mu(x)U_\mu(x + \mu)U_\mu(x + 2\mu) \times \dots \times U_\mu(x + (L - 1)\mu). \quad (4)$$

Under the above symmetry,  $P_\mu(x)$  gets multiplied by a phase. The gauge invariant content of  $P_\mu(x)$  is its set of eigenvalues (the spectrum)  $e^{i\theta_i^P}$ ,  $i = 1, 2, \dots, N$ . The ordering is not gauge invariant, and there is a constraint that  $\det P_\mu(x) = 1$ . Under a  $Z(N)$  transformation, the set of eigenvalues is circularly shifted by a fixed amount. The spectra of  $P_\mu(x)$  and of  $P_\mu(x + j\mu)$  are the same for all  $j = 0, 1, 2, \dots, L - 1$ . Wilson loops are defined similarly to Polyakov loops, only they are invariant under the  $Z^4(N)$ . Often we shall speak about the angles  $\theta_i^P$ , thinking about them round a circle and referring to them also as the "spectrum".

At a given  $L$ , we increase  $b$  gradually, until a point is reached where one of the four  $Z(N)$  factors, acting in a randomly picked direction  $\mu$ , breaks spontaneously. The breaking is reflected by a change in the spectra of  $P_\mu(x)$  away from a form symmetric under circular shifts. This happens when a gap larger than  $\frac{2\pi}{N}$  opens up in the angle spectrum at some random location round the circle. This event can be detected in various ways.

At infinite  $N$ , six phases are encountered as  $b$  is varied from zero to infinity on a lattice of size  $L^4$  so long as  $L \geq 9$ . In the range  $0 < b < 0.36$  the system is in a “hot” phase (denoted by “0h”), where the  $Z^4(N)$  is preserved and the  $1 \times 1$  Wilson loop has no gap in its spectrum. As  $b$  increases one goes into a “0c” phase, where the  $Z^4(N)$  symmetry still is preserved, but the  $1 \times 1$  Wilson loop now has a gap in its spectrum. The gap is centered at the point  $-1$  on the unit circle, so that charge conjugation is also preserved. Next one goes into a “1c” phase, where exactly one factor of the global  $Z^4(N)$  is broken. As  $b$  increases further, additional factors of  $Z^4(N)$  successively break until the phase “4c” is reached, which extends all the way to  $b = \infty$ . For  $5 \leq L \leq 8$  the phases “0c” and “1c”, including the transition between them, can be extended downwards in  $b$ , as metastable phases, into the “0h” phase region. Thus, using metastability, we can extend most of the phase structure of interest from  $L \geq 9$  to  $L \geq 5$ . For  $L \leq 4$  the stable “0c” phase is “squeezed” out, and we are left with only five phases. The case  $L = 1$  is the original Eguchi–Kawai model. All this holds with Wilson’s single plaquette action and might change with a different lattice action. However, physical properties that survive the continuum limit should be insensitive to the choice of action.

The “0c” phase is the most interesting phase because there planar QCD exhibits confinement and stringy behavior at large distances and field theoretic asymptotic freedom at short distances. The phase “4c” is the phase where planar QCD is well described by Bjorken’s femto-universe heated to high temperature.

There is little doubt that the “4c” phase survives in the continuum limit. This means that there exists a finite range of torus sides  $l$  between zero and some small scale where planar continuum QCD is in a “4c” phase. There also is little doubt that the “0h” phase does not have a continuum limit, i.e., there is no finite

range of  $l$ -values in which the continuum system is in a “0h” phase. In other words, the “0h” phase is a lattice artifact. In this Letter, we shall present evidence that the “0c” phase does have a continuum limit, describing the system in the range  $\infty > l > l_c$ . More work will be needed to complete the continuum phase diagram and see in detail how the system goes from the “0c” phase all the way to the “4c” phase as the torus is shrunk in size.

### 3. Numerical method and results

We simulated the system using the Monte Carlo method employing heat bath updates and overrelaxation updates. The heat-bath updates amounted to sequential  $SU(2)$  updates, going over a set of  $\frac{N(N-1)}{2}$  of  $SU(2)$  subgroups identified by choosing two distinct integers between 1 and  $N$  [6]. For most of the values of  $b$  we used, each  $SU(2)$  update was done using the Kennedy–Pendelton method [8]. For few small values of  $b$  we used the original Creutz method [7]. The cost of a heat-bath update goes as  $N^3$  as  $N$  increases.

The overrelaxation update was a full  $SU(N)$  update [9–11] and had a comparable cost. Our implementation went as follows.

The portion of the action  $S$  that depends on a particular link matrix, denoted by  $U$ ,  $S_R(U)$ , is given by

$$\text{Tr}[U\Sigma] = \frac{1}{2} [S_R(U) + iS_I(U)] \quad (5)$$

with real  $S_{R,I}(U)$ .  $\Sigma$  are the “staples”, a positive number (the coupling) times a sum of simple unitary matrices.  $\Sigma$  is determined by  $U$  and when this is not evident from the context we shall use the notation  $\Sigma_U$ . With probability one,  $\Sigma$  has nonzero determinant permitting a unique definition of a unitary matrix  $V_\Sigma$ :

$$V_\Sigma = \frac{1}{\sqrt{\Sigma \Sigma^\dagger}} \Sigma, \quad \det V_\Sigma = \frac{\det \Sigma}{|\det \Sigma|} \equiv e^{i\Phi_\Sigma}. \quad (6)$$

$V_\Sigma$  is calculated as follows:  $\Sigma \Sigma^\dagger$  and  $\Sigma^\dagger \Sigma$  are diagonalized using the Householder method. The eigenvalues are distinct with probability one and make up a positive diagonal matrix  $\mathbf{D}$ . The diagonalizing matrices provide two representations of  $\mathbf{D}$ :  $\mathbf{D} =$

$X \Sigma^\dagger \Sigma X^\dagger = Y \Sigma \Sigma^\dagger Y^\dagger$  and, finally, we end up with  $V_\Sigma = Y^\dagger X$ .

Taking  $\Phi_\Sigma$  to obey  $\pi \geq \Phi_\Sigma > -\pi$  we define a new  $SU(N)$  matrix by

$$V = e^{\frac{2i\Phi_\Sigma}{N}} V_\Sigma^\dagger U^\dagger V_\Sigma^\dagger. \quad (7)$$

The update starts by “offering” the replacement  $U \rightarrow V$ . The new action is given by

$$S_R(V) = \cos\left(\frac{2\Phi_\Sigma}{N}\right) S_R(U) + \sin\left(\frac{2\Phi_\Sigma}{N}\right) S_I(U). \quad (8)$$

For large  $N$  we expect  $S_R(U)$  to be order  $N^2$  and  $S_I(U)$  to be order  $N$ . This implies that the change in action is order one and hence there is an order one probability of making a large move in configuration space. In the Metropolis step, the a priori probability for change is unity for the  $U \rightarrow V$  transition and zero for anything else. Applied twice, this transition becomes the identity; therefore the Metropolis step only depends on  $R$ , the ratio of the Boltzmann factors.

$$R = \exp\left\{\sin\left(\frac{2\Phi_\Sigma}{N}\right) S_I(U) - \left[1 - \cos\left(\frac{2\Phi_\Sigma}{N}\right)\right] S_R(U)\right\}. \quad (9)$$

The acceptance probability for the change is taken as  $\min\{1, R\}$ ; this satisfies detailed balance.

We found that the acceptance rate for the overrelaxed update was over 95% for all our  $N$  and  $b$  values. We employed a mixture of heat bath and overrelaxation steps of equal amounts. A more comprehensive independent study of full  $SU(N)$  overrelaxation has been recently presented in [11].

Our Polyakov loops (as well as various Wilson loops we looked at) were built out of  $\tilde{U}_\mu(x)$  matrices, rather than the original link matrices  $U_\mu(x)$ . The  $\tilde{U}_\mu(x)$  matrices are defined in term of the  $U_\mu(x)$  by an iterative “smearing” procedure [12]. One step in the iteration takes one from a set  $U_\mu^{(n)}(x)$  to a set  $U_\mu^{(n+1)}(x)$ , by the following equation:

$$\begin{aligned} X_\mu^{(n+1)}(x) &\equiv \alpha U_\mu^{(n)}(x) + \frac{1-\alpha}{6} \Sigma_{U_\mu^{(n)}(x)}, \\ U_\mu^{(n+1)}(x) &= X_\mu^{(n+1)}(x) \frac{1}{\sqrt{[X_\mu^{(n+1)}(x)]^\dagger X_\mu^{(n+1)}(x)}}. \end{aligned} \quad (10)$$

We chose  $\alpha = 0.45$  and iterated  $L$ -times:

$$\tilde{U}_\mu(x) = U_\mu^{(L)}(x). \quad (11)$$

This has little effect on Polyakov spectra at smaller  $b$  values, but, after the transition the well-known ultraviolet renormalization [13] of Polyakov loops will reduce all traces  $\text{Tr } P_\mu^k(x, \mu)$ , effectively making the angle spectrum look more uniform and making the transition harder to discern. This effect is reduced by the above smearing. In our three-dimensional work [4] we could do without smearing, since the ultraviolet divergence is milder. The Polyakov loops in terms of the smeared links have the same symmetry properties as the Polyakov loops in terms of the original links and therefore provide perfectly adequate order parameters for  $Z(N)$  symmetry breaking.

We looked at several observables. Two were the most useful for identifying the phase transition. The first is [24]

$$p(\tilde{P}_\mu) = \frac{1}{N^2} \left\langle \sum_{i,j=1}^N \sin^2 \frac{1}{2} (\theta_i^{\tilde{P}} - \theta_j^{\tilde{P}}) \right\rangle. \quad (12)$$

The averaging is over the 3-plane perpendicular to  $\mu$  and over configurations. Equally spaced angles respect the  $Z(N)$  symmetry in the  $\mu$  direction and maximize  $p$  to 0.5. When the angle-spectrum starts getting modulated and opens a gap,  $p$  drops below 0.5. The second observable that we found useful,  $\rho(s_{\max})$ , is constructed as follows: among all spacing between adjacent angles round the circle select the largest one,  $s_{\max}$ . Its probability distribution,  $\rho(s_{\max})$ , strongly depends on whether the  $Z(N)$  associated with the direction under consideration is broken or not. When the  $Z(N)$  is not broken the distribution just reflects universal angle repulsion. However, when a gap opens in the spectrum, it dominates the distribution. Of course, looking directly at the histograms of the angles associated with individual directions remains the most direct way to observe the behavior of the system. Figs. 1 and 2 show examples of the evolution of  $p$  when the run started from a typical configuration in the wrong phase. Fig. 3 shows an example of the difference between the distribution of the maximal level spacing in the direction corresponding to a broken  $Z(N)$  and the maximal level spacing distributions in the other directions, whose corresponding  $Z(N)$ 's are unbroken.

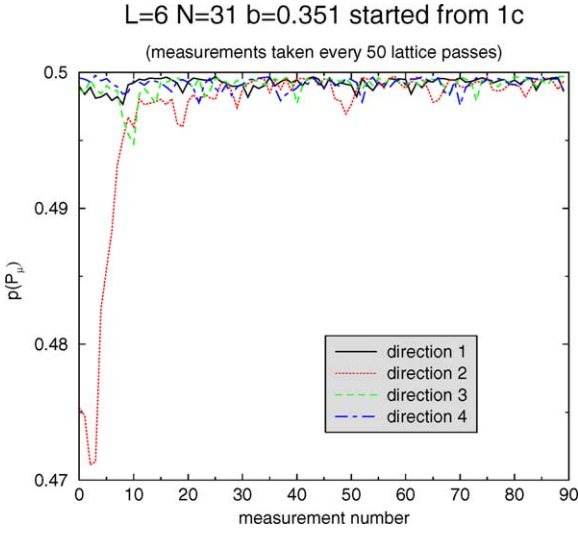


Fig. 1. History of the variable  $p(\tilde{P}_\mu)$  for each direction. We see the evolution from a state where one of the four  $Z(N)$  factors is broken to one in which all four are preserved.

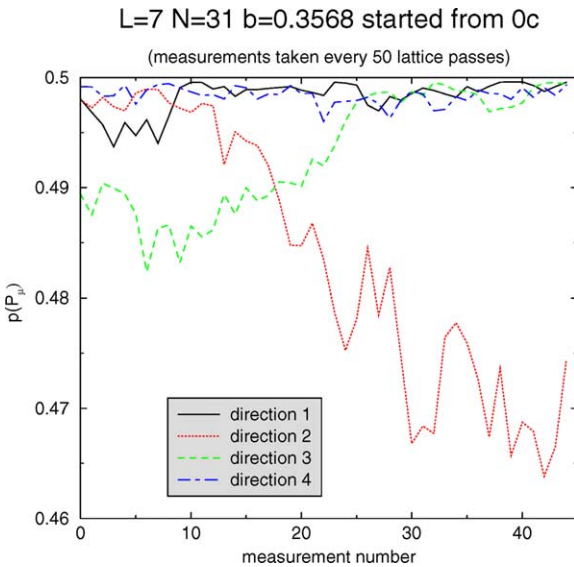


Fig. 2. History of the variable  $p(\tilde{P}_\mu)$  for each direction. We see the evolution from a state where all four  $Z(N)$  factors are preserved to one where one factor is broken. During the first fifty passes (before the first measurement) Polyakov loops in direction 3 have acquired some structure but, ultimately, direction 2 is selected for breakdown and the Polyakov loops in the other three directions converge to a symmetric state.

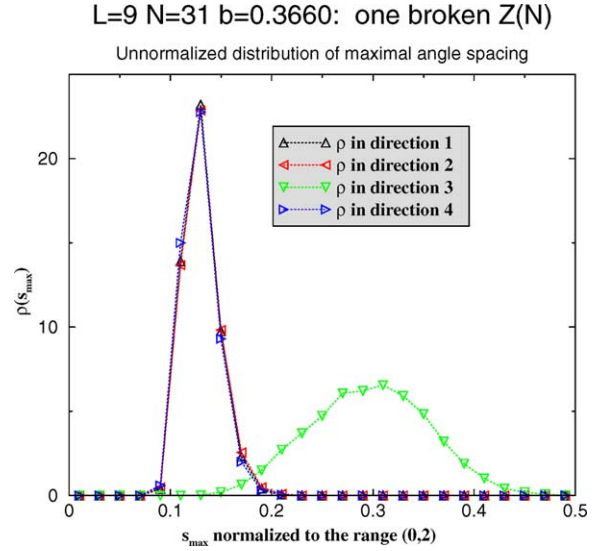


Fig. 3. Here we show the difference between the distributions of the largest inter-angle spacing for smeared Polyakov loops in different directions in the phase where exactly one  $Z(N)$  factor is broken. (At other couplings, where no  $Z(N)$  factor is broken, all four distributions look like the three unbroken ones here.)

Fig. 4 shows an example of the angle distributions associated with the four independent directions when the system is in the “1c” phase. The superposed oscillations reflect the global  $SU(N)$  constraint on angle values. For a  $SU(N)$  matrix drawn with Haar probability measure a relatively straightforward calculation gives, with  $\xi = \frac{\theta^P}{\pi}$ :

$$p(\xi) d\xi = \left[ \frac{1}{2} + \frac{(-1)^{N-1}}{N} \cos(N\pi\xi) \right] d\xi. \quad (13)$$

Thus, the expected swing between minima and maxima should be  $\frac{2}{27}$  in our plot and this fits more or less. Also, because 27 is an odd number, we have a maximum, rather than a minimum, at  $\xi = 0$ .

The “0c” to “1c” transition also breaks hypercubic invariance since one direction is randomly selected by the breaking. The “0c” to “1c” is transition most likely is of first order, and therefore generates hysteresis cycles for relatively short runs. Our runs were of the order of few thousands and did not allow a very precise identification of the location of the transition or a definitive determination of its order. We ran hysteresis cycles looking for two extremes which determine the range we believe the true transition is in. At the first extreme we start from a configuration

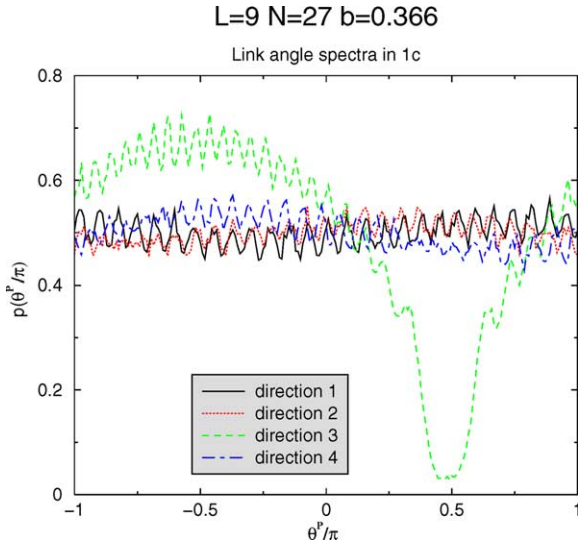


Fig. 4. Angle distributions in four directions in the “1c” phase. There are twenty seven periods in the superposed oscillations. The peaks, except close to the gap associated with direction 3, are equally spaced.

typical of the “1c” phase, and see that the system disorders, restoring the remaining  $Z(N)$  factor. This is what happens in Fig. 1. For our larger volumes ( $L = 9$  and  $L = 10$ ) we try to find the largest  $b$  where the configuration evolves in this way within at most 3000–4000 passes over the lattice. At the other extreme we start from a totally symmetric configuration and observe the system evolving into a “1c” phase, like in Fig. 2. Here we try to find the smallest  $b$  where this scenario is realized. The ranges in  $b$  so obtained were of length somewhere between  $1 \times 10^{-3}$  and  $3 \times 10^{-3}$ . A few checks, performed by varying  $N$  at fixed  $L$ , showed that the finite  $N$  effects were at most of order  $5 \times 10^{-4}$ . With this accuracy, we were able to check whether the location of the transition  $b_c(L)$  varies with  $L$  in a way compatible with asymptotic freedom. Our numerical work used up about one year’s worth of time on a dedicated desktop PC with a modern processor and 2GB of memory. With this rough map of the phases in place, one could proceed to finer determinations, but this would require one or two orders of magnitude more computer time.

If the transition we are searching for truly is a continuum phenomenon, the inverse of the function

$b_c(L)$ ,  $L_c(b)$ , should behave for  $b \rightarrow \infty$  as

$$L_c(b) \sim L_0 \left( \frac{11}{48\pi^2 b} \right)^{51/121} e^{24\pi^2 b/11}. \quad (14)$$

The asymptotic regime is not reached at  $L \sim 10$ , but by the “tadpole” [17] replacement

$$b \rightarrow b_I \equiv b e(b), \quad e(b) = \frac{1}{N} \langle \text{Tr } U_{\mu, \nu}(x) \rangle, \quad (15)$$

$$L_c(b) \sim L_0^I \left( \frac{11}{48\pi^2 b_I} \right)^{51/121} e^{24\pi^2 b_I/11} \quad (16)$$

the asymptotic behavior is supposed to set in much earlier [18]. The numerical effect the replacement of  $b$  by  $b_I$  has is summarized by the approximate relation  $\delta b_I \sim 1.3 \delta b$  which holds in the vicinity of the transition at  $L = 9$ . With  $b$  replaced by  $b_I$ , the theoretical curve becomes somewhat steeper. The plaquette expectation value,  $e(b)$ , is taken on the symmetric side of the transition and using the MC data, can be well fitted in the range of interest by

$$e(b) \approx \frac{1 + \frac{a_0}{b} + \frac{a_1}{b^2}}{1 + \frac{a_2}{b} + \frac{a_3}{b^2}}. \quad (17)$$

When  $b \rightarrow \infty$ ,  $e(b) = 1$  at leading order in  $\frac{1}{b}$  and  $b_I = b$ . We varied  $L$  between 4 and 10 and  $b$  between 0.344 and 0.366.  $e(b)$  was reasonably well fit by  $a_0 = -0.58964$ ,  $a_1 = 0.08467$ ,  $a_2 = -0.50227$ ,  $a_3 = 0.05479$ . Hence,

$$L_0 = L_0^I e^{24\pi^2(a_0 - a_2)/11} \approx 0.1524 L_0^I. \quad (18)$$

The ranges we determined for the “0c” to “1c” transitions are reasonably well described by a range of  $L_0^I$  constants between 0.245 and 0.275. Fig. 5 shows the ranges we established on a plot together with lines representing the tadpole improved two loop renormalization formula with different amplitudes. The relative consistency of this fit constitutes our numerical evidence that the transition is physical, occurring in the continuum at a finite scale. The relevant numbers for the “0c” to “1c” transition ranges that went into Fig. 5 are collected in Table 1.

Let us first discuss all the stable phases, ignoring the metastable ones. In addition to the “0c” to “1c” transition we have been focusing on, the system also undergoes a lattice transition from “0h” to an “Xc”. For small volumes, X will be larger than 0, but, starting with  $L = 9$ ,  $X = 0$ . The “0h” to “Xc” ( $X \neq 0$ )

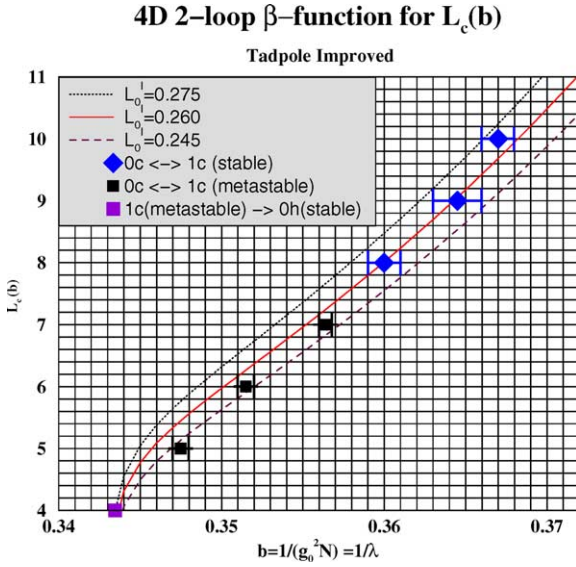


Fig. 5. The transition ranges compared to possible two loop renormalization group curves with tadpole improvement.

Table 1  
Summary of ranges for the “0c” to “1c” transitions

$L$	$N$	$(b_{\min}, b_{\max})$
5	31	(0.3470, 0.3480)
5	41	(0.3473, 0.3485)
6	31	(0.3510, 0.3520)
7	31	(0.3560, 0.3568)
8	23	(0.3590, 0.3610)
8	31	(0.3595, 0.3605)
9	23	(0.3630, 0.3655)
9	27	(0.3635, 0.3650)
9	31	(0.3630, 0.3660)
10	23	(0.3662, 0.3678)

transitions at smaller  $L$  values are strongly first order and occur at  $b = b_{\text{BULK}}$ . A similar transition also occurs at  $L = \infty$  for any  $N \geq 5$  [14]. In this case this bulk transition is well known to be associated with a large jump in  $e(b)$  at  $b_{\text{BULK}}$ . At finite  $N$  the transition does not break any symmetry.

At  $N = \infty$ , and any finite but large enough  $L$ , the location of the bulk transition can be estimated with the help of [15] to occur at  $b_{\text{BULK}} = 0.3600$ . (For an earlier determination, see [16].) So long as  $L > 8$ , the bulk transition again breaks no symmetry: it takes the system from the “0h” phase to the “0c” phase. Again,  $e(b)$  undergoes a significant jump at  $b_{\text{BULK}}$ .

In addition, with  $N = \infty$ , the average eigenvalue distribution of the plaquette variable now undergoes also a qualitative change, opening a gap at angle  $\pi$  when  $b$  increases through  $b_{\text{BULK}}$ .

We have investigated the  $L = 1$  case in some detail. This case is special, and the algorithms we used are somewhat different; since this is a bit of a side issue, we shall not elaborate in detail. We found evidence for five stable phases. Examples of firmly determined couplings in each phase are:  $b = 0.150$  in “0h”,  $b = 0.205$  in “1c”,  $b = 0.235$  in “2c”,  $b = 0.275$  in “3c” and  $b = 0.320$  in “4c”. We also found approximate locations for the transitions: the “0h” to “1c” transition occurs at  $b = 0.19$ , the “1c” to “2c” transition occurs at  $b = 0.22$ , the “2c” to “3c” transition occurs at  $b = 0.26$  and the “3c” to “4c” transition occurs at  $b = 0.30$ .

It is easy to keep the system in “cold” (“Xc”) phases even for  $b < b_{\text{BULK}}$ . These phases are in principle metastable. However, in practice, for large enough  $N$ , ( $N \geq 20$ ), the “Xc” phases are very stable in a Monte Carlo simulation. This makes it possible to investigate the “0c” to “1c” transitions of interest also for  $L \leq 7$ . All our values for  $b_c(L)$  for  $L \leq 7$  have  $b_c(L) < b_{\text{BULK}}$ .

#### 4. Relation to the finite temperature deconfinement transition

Suppose we studied a torus of unequal sides,  $L_\mu$ . The most plausible assumption is that again one  $Z(N)$  will break first as  $b$  is increased from the phase where the entire  $Z^4(N)$  is preserved. Only now, which  $Z(N)$  breaks will no longer be arbitrary, but, rather, the one associated with the direction with the shortest  $L_\mu$  is selected to break first. Call this direction  $\mu_0$ . The arguments from our previous paper [4] say that for  $b$ ’s smaller than this transition point there is no dependence on the parameters  $L_\mu$ . Hence, up to the transition, all  $L_\mu$ ’s can be considered as infinite. But, we could equally well think about the  $L_\mu$  with  $\mu \neq \mu_0$  as infinite, while keeping in mind that  $L_{\mu_0}$  is finite. This puts the system in a situation considered in [19]. Then, there would be a transition as  $b$  is increased, even for finite  $N$ . This would be a finite temperature transition, which is first order at large  $N$ , and has a finite limit at  $N = \infty$ ,  $T_c$  [18]. The simplest



consistent assumption is that  $l_c = \frac{1}{T_c}$ . This is in full agreement with the viewpoint of the authors of [20].

In Ref. [18] the string tension in lattice units is found to behave approximately as follows:

$$\frac{1}{\sqrt{\sigma}} \sim \frac{1}{\sqrt{\sigma_0^l}} \left( \frac{11}{48\pi^2 b_l} \right)^{51/121} e^{24\pi^2 b_l / 11}. \quad (19)$$

With a simple minded extrapolation to  $N = \infty$ , we obtain from [18]  $\sqrt{\sigma_0^l} \approx 6.05$ . Combining this with our value of  $L_0^l \approx 0.26$  we find  $\frac{1}{L\sqrt{\sigma}} \approx 0.64$ . The most up to date value for the infinite  $N$  value of  $\frac{T_c}{\sqrt{\sigma}}$  can be found in [20]. It is about 0.60. Thus,  $l_c = \frac{1}{T_c}$  is consistent with what we know to date, but the evidence is not compelling.

The special physical effects surrounding the finite temperature transition in pure YM in the planar limit were first discussed in [21]. More aspects have been studied in [22]. Earlier numerical studies of the infinite  $N$  finite temperature transition in  $SU(N)$  gauge theories can be found in [23].

## 5. Preserving $l$ independence in the meson sector

We have emphasized the volume independence of the pure gauge theory in the large  $l$  phase. It is natural to ask whether fermions moving in the gauge backgrounds typical to this phase also will behave as if the volume were infinite. This question needs to be sharpened because we are considering here only a finite number of flavors, which makes the fermions “quenched” as a result of the large color ( $N$ ) limit. The fermions simply provide definitions for particular nonlocal gauge invariant observables, but do not influence the distribution of the gauge background. The  $l$  independence holds only for single traces of Wilson loops, and there is a way even for a very large loop to fold up into the  $l^4$  torus. But a trace of the product of a fermion by an antifermion propagator only depends on the end points, not on a path, so there seems to be no way to describe a separation that would not fit into the torus.

However, there is a trick which seems to allow the definition of fermionic observables on the finite torus which nevertheless describe propagation at larger distances. This trick is at the heart of our proposed short-

cut to the planar limit in the meson sector. It is a simple generalization of work in [24,25]. This produces a prescription for calculating  $q\bar{q}$  properties in the large  $N$  limit while preserving volume independence. At the diagrammatic level it is easy to understand what we have done [25], but, to be sure, the arguments supporting this construction are far from rigorous. For this reason, we have undertaken an extensive test in two dimensions [26], which came out favorable. We believe that this provides sufficient grounds to go ahead and see what happens in four dimensions.

To be concrete, let us consider the scalars  $M(x) = \frac{1}{\sqrt{N}} \bar{\psi} \chi(x)$  and  $\bar{M}(x) = \frac{1}{\sqrt{N}} \bar{\chi} \psi(x)$ . These meson fields are color singlets.  $x$  and  $y$  are sites on an  $L^4$  lattice. Normally we would expect only momenta  $K_\mu = \frac{2\pi}{L} k_\mu$  with  $k_\mu = 0, 1, \dots, L-1$  to be accessible. We claim that large  $N$  reduction makes it possible to interpret data obtained on the  $L^4$  lattice as providing predictions for momenta on an  $(NL)^4$  lattice, at leading order in the  $\frac{1}{N}$  expansion. The momenta are now written as  $K + Q$  where  $K$  is as before, and  $Q_\mu = \frac{2\pi}{NL} q_\mu$  with  $q_\mu = 0, 1, \dots, N-1$ . We are after an expression for the meson–meson propagator,  $S(K + Q)$ . We first define a shifted link field  $U_\mu^{(q)}(x)$  by

$$U_\mu^{(q)}(x) = e^{\frac{2\pi i}{NL} q_\mu} U_\mu(x) \quad (20)$$

and denote the fermion  $\psi - \bar{\psi}$  and  $\chi - \bar{\chi}$  propagators on the lattice, in a given gauge background,  $\{U\}$ , by  $G(x, y, \{U\})$ . The shifted gauge field links are not in  $SU(N)$ . Let us consider the collection of all Wilson and Polyakov loops made out of the  $U_\mu^{(q)}(x)$  variables as elementary links. Before taking the trace all the unitary matrices are back in  $SU(N)$  and could have been obtained from elementary links that also are all in  $SU(N)$ . This works because our observables are gauge invariant even under  $U(1)$  gauge transformations that take the links out of  $SU(N)$ . The dependence on the integers  $q_\mu$  is removable by a  $Z^4(N)$  symmetry transformation. Thus, so long as we treat a fermionic observable that depends on a single gauge field background, there is no dependence on  $q_\mu$  so long the  $Z^4(N)$  symmetry is not broken. In other words, we have to be in the “0c” phase. The  $q_\mu$  assume their role as momentum “gap fillers” only when we consider an observable that depends on different gauge field backgrounds. The Dirac and color indices of  $G(x, y, \{U\})$



are suppressed. We now define the quantity

$$R(x, y; \{U\}, q) = -\frac{1}{N} \text{Tr}[G(x, y, \{U^{(q)}\})G(y, x, \{U^{(0)}\})]. \quad (21)$$

The trace in the above equation sums over color and Dirac indices. The key formula for  $S(K + Q)$  is given by

$$S(K + Q) = \sum_x e^{\frac{2\pi i}{L}k(x-y)} \langle R(x, y; \{U\}, q) \rangle_{\{U\}}. \quad (22)$$

The averaging over  $\{U\}$  restores translational invariance.

If we are not in the “0c” phase the procedure fails. One could try to impose a “0c” phase by quenching the links on a  $1^4$  lattice, but implementing the additional averaging puts too big a burden on the numerics and is likely less practicable than the procedure we propose here.

## 6. Conclusions

Our previous work [4] and the present Letter make a plausible case for the following scenario: planar QCD on a torus of side  $l$  has a nontrivial phase structure as a function of  $l$ . When  $l$  decreases from  $\infty$  to any  $l < l_c$ , the system undergoes a phase transition where the global  $Z^4(N)$  symmetry breaks spontaneously. It is likely that  $l_c = \frac{1}{T_c}$  where  $T_c$  is the infinite  $N$  limit of the finite temperature deconfinement transitions of  $SU(N)$  YM theory at finite  $N$ 's. The precise determination of  $l_c$ , including error estimates, and of the existence and locations of other continuum transitions needs substantially more numerical effort than invested to date, but is a feasible numerical project.

A distinctive property of the  $l > l_c$  phase is the  $l$ -independence of arbitrary Wilson loops, which provides a continuum realization of lattice Eguchi–Kawai reduction. While the free energy does not depend on  $l$  at leading order in a generic, free, string theory with toroidal target space, among field theories, only certain gauge theories in the planar limit can exhibit such a property. This scenario, in turn, leads to the conjecture that at  $l > l_c$  the  $l$  dependence of meson propagators can also be made to disappear at leading order in the  $\frac{1}{N}$  expansion by quenching, providing the oppor-

tunity for a numerical shortcut to the infinite  $N$  meson sector of  $SU(N)$  YM theory in infinite space–time.

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